The Gröbner basis for powers of a general linear form in a monomial complete intersection

Filip Jonsson Kling[1], Samuel Lundqvist[1], Fatemeh Mohammadi[2],
Matthias Orth[2]

[1]Stockholm University, Sweden [2]KU Leuven, Belgium

MEGA: Effective Methods in Algebraic Geometry
Durham, 6 July 2026

Table of Contents

Standard graded algebras and monomial ideals

We work with *standard graded algebras*, of the form

$$\mathbf{k}[x_1, \ldots, x_n]/I,$$

where

Standard graded algebras and monomial ideals

We work with *standard graded algebras*, of the form

$$\mathbf{k}[x_1, \ldots, x_n]/I,$$

where

- the field $\mathbf{k}$ is of characteristic zero,

Standard graded algebras and monomial ideals

We work with *standard graded algebras*, of the form

$$\mathbf{k}[x_1, \ldots, x_n]/I,$$

where

- the field $\mathbf{k}$ is of characteristic zero,
- all variables x_i of the polynomial ring $R = \mathbf{k}[x_1, \ldots, x_n]$ have degree 1,

Standard graded algebras and monomial ideals

We work with *standard graded algebras*, of the form

$$\mathbf{k}[x_1, \ldots, x_n]/I,$$

where

- the field $\mathbf{k}$ is of characteristic zero,
- all variables x_i of the polynomial ring $R = \mathbf{k}[x_1, \ldots, x_n]$ have degree 1,
- and the relation ideal I is homogeneous.

Standard graded algebras and monomial ideals

We work with *standard graded algebras*, of the form

$$\mathbf{k}[x_1, \ldots, x_n]/I,$$

where

- the field $\mathbf{k}$ is of characteristic zero,
- all variables x_i of the polynomial ring $R = \mathbf{k}[x_1, \ldots, x_n]$ have degree 1,
- and the relation ideal I is homogeneous.

We fix a monomial order $\prec$. Then,

- every non-zero polynomial of R has a leading monomial,

Standard graded algebras and monomial ideals

We work with *standard graded algebras*, of the form

$$\mathbf{k}[x_1, \ldots, x_n]/I,$$

where

- the field $\mathbf{k}$ is of characteristic zero,
- all variables x_i of the polynomial ring $R = \mathbf{k}[x_1, \ldots, x_n]$ have degree 1,
- and the relation ideal I is homogeneous.

We fix a monomial order $\prec$. Then,

- every non-zero polynomial of R has a leading monomial,
- and every non-zero ideal $I \subseteq R$ has
  - an initial monomial ideal $\mathrm{in}_\prec(I)$
  - a unique reduced $\prec$-Gröbner basis.

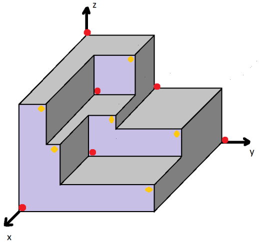


Figure: *Escalier* of a monomial ideal.

Hilbert series

Definition

Let $M = \oplus_{d \geq 0} M_d$ be a finitely generated graded R-module with degree-d components M_d. The *Hilbert series* of M is

$$\mathrm{HS}(M;t) = \sum_{d=0}^{\infty} \dim_{\mathbf{k}}(M_d)\, t^d\,.$$

Hilbert series

Definition

Let $M = \oplus_{d\geq 0} M_d$ be a finitely generated graded R-module with degree-d components M_d. The *Hilbert series* of M is

$$\mathrm{HS}(M;t) = \sum_{d=0}^{\infty} \dim_{\mathbf{k}}(M_d)\, t^d\,.$$

Example

For $A = \mathbf{k}[x_1, x_2]/(x_1^2, x_2^3)$, we have $\mathrm{HS}(A;t) = 1 + 2t + 2t^2 + t^3$.

Hilbert series

Definition

Let $M = \oplus_{d \geq 0} M_d$ be a finitely generated graded R-module with degree-d components M_d. The *Hilbert series* of M is

$$\mathrm{HS}(M; t) = \sum_{d=0}^{\infty} \dim_{\mathbf{k}}(M_d)\, t^d \,.$$

Example

For $A = \mathbf{k}[x_1, x_2]/(x_1^2, x_2^3)$, we have $\mathrm{HS}(A; t) = 1 + 2t + 2t^2 + t^3$.

Remark

Let $\{0\} \neq I \subseteq R$ be a homogeneous ideal, and $\prec$ a monomial ordering compatible with degrees. Then, $\mathrm{HS}(R/I; t) = \mathrm{HS}(R/\operatorname{in}_{\prec}(I); t)$.

Open problems on Hilbert series

Given homogeneous ideal $I \subseteq R = \mathbf{k}[x_1, \dots, x_n]$ and homogeneous element $h \in R \setminus I$ of degree d, is it true that if the generators of I and h are generic of prescribed degrees, then all the multiplication maps $\cdot h : (R/I)_e \to (R/I)_{e+d}$ must have maximal rank?

This is unknown for I generated by more than n elements!

Open problems on Hilbert series

Given homogeneous ideal $I \subseteq R = \mathbf{k}[x_1, \dots, x_n]$ and homogeneous element $h \in R \setminus I$ of degree d, is it true that if the generators of I and h are generic of prescribed degrees, then all the multiplication maps $\cdot h : (R/I)_e \to (R/I)_{e+d}$ must have maximal rank?

This is unknown for I generated by more than n elements!

Fröberg's conjecture (1985)

The Hilbert series for the quotient of R by m general forms of degrees $a_1, \dots, a_m$ is

$$\left[\frac{\prod_{i=1}^{m}(1 - t^{a_i})}{(1 - t)^n}\right],$$

where $[\dots]$ denotes truncation at the first non-positive coefficient.

Open problems on Hilbert series

Given homogeneous ideal $I \subseteq R = \mathbf{k}[x_1, \ldots, x_n]$ and homogeneous element $h \in R \setminus I$ of degree d, is it true that if the generators of I and h are generic of prescribed degrees, then all the multiplication maps $\cdot h : (R/I)_e \to (R/I)_{e+d}$ must have maximal rank?

This is unknown for I generated by more than n elements!

Fröberg's conjecture (1985)

The Hilbert series for the quotient of R by m general forms of degrees $a_1, \ldots, a_m$ is

$$\left[\frac{\prod_{i=1}^{m}(1 - t^{a_i})}{(1 - t)^n}\right],$$

where $[\ldots]$ denotes truncation at the first non-positive coefficient.

Overarching goal

Understand graded ideals and algebras in generic position or defined by generic forms, using constructive methods of Computer Algebra.

A generic object

Definition

We study $I_{n,\mathbf{m},k} = (x_1^{m_1}, \dots, x_n^{m_n}, (x_1 + \cdots + x_n)^k) \subset \mathbf{k}[x_1, \dots, x_n]$, where $\mathrm{char}(\mathbf{k}) = 0$, $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 2}$ and $k \geq 1$.

A generic object

Definition

We study $I_{n,\mathbf{m},k} = (x_1^{m_1}, \ldots, x_n^{m_n}, (x_1 + \cdots + x_n)^k) \subset \mathbf{k}[x_1, \ldots, x_n]$, where $\mathrm{char}(\mathbf{k}) = 0$, $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 2}$ and $k \geq 1$.

- These ideals and their algebras can be regarded as generic because they attain the Hilbert series of Fröberg's conjecture (Stanley 1980; Watanabe 1989).

A generic object

Definition

We study $I_{n,\mathbf{m},k} = (x_1^{m_1}, \ldots, x_n^{m_n}, (x_1 + \cdots + x_n)^k) \subset \mathbf{k}[x_1, \ldots, x_n]$, where $\mathrm{char}(\mathbf{k}) = 0$, $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 2}$ and $k \geq 1$.

- These ideals and their algebras can be regarded as generic because they attain the Hilbert series of Fröberg's conjecture (Stanley 1980; Watanabe 1989).
- These *almost complete intersections* have attracted much attention recently.
  - Booth, Singh, and Vraciu, preprint 2024
  - Diethorn, Güntürkün, Hardesty, Mete, Şega, Sobieska, preprint 2025
  - Jonsson Kling, Lundqvist, Mohammadi, O., Sáenz-de-Cabezón, 2026
  - Jonsson Kling, Lundqvist, Mohammadi, O., 2026

A generic object

Definition

We study $I_{n,\mathbf{m},k} = (x_1^{m_1}, \ldots, x_n^{m_n}, (x_1 + \cdots + x_n)^k) \subset \mathbf{k}[x_1, \ldots, x_n]$, where $\mathrm{char}(\mathbf{k}) = 0$, $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 2}$ and $k \geq 1$.

- These ideals and their algebras can be regarded as generic because they attain the Hilbert series of Fröberg's conjecture (Stanley 1980; Watanabe 1989).
- These *almost complete intersections* have attracted much attention recently.
  - Booth, Singh, and Vraciu, preprint 2024
  - Diethorn, Güntürkün, Hardesty, Mete, Şega, Sobieska, preprint 2025
  - Jonsson Kling, Lundqvist, Mohammadi, O., Sáenz-de-Cabezón, 2026
  - Jonsson Kling, Lundqvist, Mohammadi, O., 2026
- Our approach is the first to obtain explicit Gröbner bases and yields new, independent proof of Stanley's and Watanabe's results.

Table of Contents

Artinian monomial complete intersections and lattice paths

Monomial ideals $P_{n,\mathbf{m}} = (x_1^{m_1}, \dots, x_n^{m_n}) \subset R$ with $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 2}$ define Artinian complete intersection algebras $A_{n,\mathbf{m}} = R/P_{n,\mathbf{m}}$ which

1. are Gorenstein,
2. have the strong Lefschetz property,
3. and have palindromic, unimodal Hilbert series.

Artinian monomial complete intersections and lattice paths

Monomial ideals $P_{n,\mathbf{m}} = (x_1^{m_1}, \ldots, x_n^{m_n}) \subset R$ with $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 2}$ define Artinian complete intersection algebras $A_{n,\mathbf{m}} = R/P_{n,\mathbf{m}}$ which

1. are Gorenstein,
2. have the strong Lefschetz property,
3. and have palindromic, unimodal Hilbert series.

Definition

A monomial $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R$ is called $\mathbf{m}$-*free* if $\alpha_i < m_i$ for $1 \leq i \leq n$.

Artinian monomial complete intersections and lattice paths

Monomial ideals $P_{n,\mathbf{m}} = (x_1^{m_1}, \dots, x_n^{m_n}) \subset R$ with $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 2}$ define Artinian complete intersection algebras $A_{n,\mathbf{m}} = R/P_{n,\mathbf{m}}$ which

1. are Gorenstein,
2. have the strong Lefschetz property,
3. and have palindromic, unimodal Hilbert series.

Definition

A monomial $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R$ is called $\mathbf{m}$-*free* if $\alpha_i < m_i$ for $1 \leq i \leq n$.

From an $\mathbf{m}$-free monomial, we obtain an $\mathbf{m}$-admissible lattice path of n steps right with slopes $1 - \alpha_i$, $1 \leq i \leq n$.

Artinian monomial complete intersections and lattice paths

Monomial ideals $P_{n,\mathbf{m}} = (x_1^{m_1}, \dots, x_n^{m_n}) \subset R$ with $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 2}$ define Artinian complete intersection algebras $A_{n,\mathbf{m}} = R/P_{n,\mathbf{m}}$ which

1. are Gorenstein,
2. have the strong Lefschetz property,
3. and have palindromic, unimodal Hilbert series.

Definition

A monomial $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R$ is called **m**-*free* if $\alpha_i < m_i$ for $1 \leq i \leq n$.

From an **m**-free monomial, we obtain an **m-admissible lattice path** of n steps right with slopes $1 - \alpha_i$, $1 \leq i \leq n$.

Lemma ([JLMO 2026])

Fix positive integers n, d and a vector $\mathbf{m} \in \mathbb{Z}^n_{\geq 2}$. Then the number of **m**-*admissible lattice paths ending at $(n, n-d)$ equals* $\dim_{\mathbf{k}}(R/P_{n,\mathbf{m}})_d$.

Number scheme and reflection points

For $1 \leq j \leq n$, let $\mathbf{m}_j = (m_1, \ldots, m_j)$. We arrange the coefficients of $\mathrm{HS}(\mathbf{k}; t)$, $\mathrm{HS}(\mathbf{k}[x_1]/P_{1,\mathbf{m}_1}; t), \ldots$ as columns in a number scheme.

				1
			1	4
		1	3	8
	1	2	4	10
1	1	2	3	8
	1	1	1	4
				1

Figure: Here, $\mathbf{m} = (3, 2, 2, 3)$, and $k = 2$. A piecewise linear function interpolates between the points located $k/2$ below the midpoints of the columns.

Number scheme and reflection points

For $1 \leq j \leq n$, let $\mathbf{m}_j = (m_1, \ldots, m_j)$. We arrange the coefficients of $\mathrm{HS}(\mathbf{k}; t)$, $\mathrm{HS}(\mathbf{k}[x_1]/P_{1,\mathbf{m}_1}; t), \ldots$ as columns in a number scheme.

$$
\begin{array}{ccccc}
 & & & & 1 \\
 & & & 1 & 4 \\
 & & 1 & 3 & 8 \\
 & 1 & 2 & 4 & 10 \\
1 & 1 & 2 & 3 & 8 \\
 & 1 & 1 & 1 & 4 \\
 & & & & 1
\end{array}
$$

Figure: Here, $\mathbf{m} = (3, 2, 2, 3)$, and $k = 2$. A piecewise linear function interpolates between the points located $k/2$ below the midpoints of the columns.

Example

Let $\mathbf{m} = (3, 2, 2, 3)$ and $t = x_1 x_3 x_4^2$. Its path ends at the point $(4, 0) = (4, n - \deg(t))$, where it touches the red line.

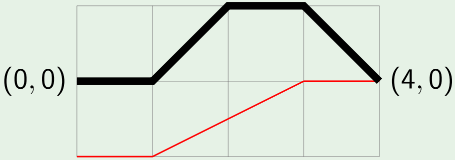


Figure: The graph of $L_{4,(3,2,2,3),2}$ and the lattice path associated with the monomial $x_1 x_3 x_4^2$.

Reflection and critical paths

Definition ([JLMO 2026])

Let $(a, b) \in \mathbb{Z}^2$ with $0 \leq a \leq n$. Reflection at the intersection of the red line and the vertical line $x = a$ yields $(a, b') =: (a, b)'$.

Reflection and critical paths

Definition ([JLMO 2026])

Let $(a,b) \in \mathbb{Z}^2$ with $0 \le a \le n$. Reflection at the intersection of the red line and the vertical line $x = a$ yields $(a,b') =: (a,b)'$. An **m**-admissible lattice path **P** with vertices $P_0 = (0,0), P_1, \ldots, P_n = (n, n-d)$ is critical if there is $0 \le i \le n$ such that

$$ {}^{i}\mathbf{P}' := (P_0, P_1, \ldots, P_{i-1}, P_i', P_{i+1}', \ldots, P_n') $$

is also **m**-admissible.

Reflection and critical paths

Definition ([JLMO 2026])

Let $(a, b) \in \mathbb{Z}^2$ with $0 \leq a \leq n$. Reflection at the intersection of the red line and the vertical line $x = a$ yields $(a, b') =: (a, b)'$. An $\mathbf{m}$-admissible lattice path $\mathbf{P}$ with vertices $P_0 = (0, 0), P_1, \ldots, P_n = (n, n - d)$ is critical if there is $0 \leq i \leq n$ such that

$$^{i}\mathbf{P}' := (P_0, P_1, \ldots, P_{i-1}, P_i', P_{i+1}', \ldots, P_n')$$

is also $\mathbf{m}$-admissible. For each critical path $\mathbf{P}$, define $\lambda(\mathbf{P})$ to be the *maximal* index i for which $^{i}\mathbf{P}'$ is $\mathbf{m}$-admissible.

Reflection and critical paths

Definition ([JLMO 2026])

Let $(a, b) \in \mathbb{Z}^2$ with $0 \le a \le n$. Reflection at the intersection of the red line and the vertical line $x = a$ yields $(a, b') =: (a, b)'$. An $\mathbf{m}$-admissible lattice path $\mathbf{P}$ with vertices $P_0 = (0, 0), P_1, \dots, P_n = (n, n - d)$ is critical if there is $0 \le i \le n$ such that

$$^{i}\mathbf{P}' := (P_0, P_1, \dots, P_{i-1}, P_i', P_{i+1}', \dots, P_n')$$

is also $\mathbf{m}$-admissible. For each critical path $\mathbf{P}$, define $\lambda(\mathbf{P})$ to be the *maximal* index i for which $^{i}\mathbf{P}'$ is $\mathbf{m}$-admissible. Define the reflection map

$$\Lambda : \{\mathbf{P} \mid \mathbf{P} \text{ critical and } \mathbf{m}\text{-admissible}\} \to \{\mathbf{P} \mid \mathbf{P}\ \mathbf{m}\text{-admissible}\},$$
$$\mathbf{P} \mapsto {}^{\lambda(\mathbf{P})}\mathbf{P}'.$$

Critical path ideal

Definition ([JLMO 2026])

Let $M_{n,\mathbf{m},k}$ be the set of monomials corresponding to critical **m**-admissible lattice paths. We write $(M_{n,\mathbf{m},k})$ for the ideal generated by $M_{n,\mathbf{m},k} + P_{n,\mathbf{m}}$.

Critical path ideal

Definition ([JLMO 2026])

Let $M_{n,\mathbf{m},k}$ be the set of monomials corresponding to critical $\mathbf{m}$-admissible lattice paths. We write $(M_{n,\mathbf{m},k})$ for the ideal generated by $M_{n,\mathbf{m},k} + P_{n,\mathbf{m}}$.

Proposition ([JLMO 2026])

Let d be the degree of an $\mathbf{m}$-free monomial and $d' = (\sum m_i) - n - d + k$. The degree-wise reflection map

$$\Lambda_d : \left\{ \begin{array}{l} \mathbf{P} \text{ critical with} \\ \text{endpoint } (n, n-d) \end{array} \right\} \to \left\{ \begin{array}{l} \mathbf{P} \text{ critical with} \\ \text{endpoint } (n, n-d') \end{array} \right\}, \quad \mathbf{P} \mapsto \Lambda(\mathbf{P}).$$

is invertible with $\Lambda_d^{-1} = \Lambda_{d'}$.

Critical path ideal

Definition ([JLMO 2026])

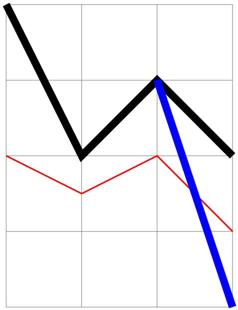
Let $M_{n,\mathbf{m},k}$ be the set of monomials corresponding to critical $\mathbf{m}$-admissible lattice paths. We write $(M_{n,\mathbf{m},k})$ for the ideal generated by $M_{n,\mathbf{m},k} + P_{n,\mathbf{m}}$.

Proposition ([JLMO 2026])

Let d be the degree of an $\mathbf{m}$-free monomial and $d' = (\sum m_i) - n - d + k$. The degree-wise reflection map

$$\Lambda_d : \left\{ \begin{array}{l} \mathbf{P} \textit{ critical with} \\ \textit{endpoint } (n, n-d) \end{array} \right\} \to \left\{ \begin{array}{l} \mathbf{P} \textit{ critical with} \\ \textit{endpoint } (n, n-d') \end{array} \right\}, \quad \mathbf{P} \mapsto \Lambda(\mathbf{P}).$$

is invertible with $\Lambda_d^{-1} = \Lambda_{d'}$.



Minimal generators and Hilbert series

We partition the set of minimal critical paths,

$$\mathrm{Crit}_{n,\mathbf{m},k} = \bigcup_{j=1}^{n} \mathrm{Crit}_{n,\mathbf{m},k,j}\,,$$

into subsets $\mathrm{Crit}_{n,\mathbf{m},k,j}$ of generators having x_j as variable of highest index.

Minimal generators and Hilbert series

We partition the set of minimal critical paths,

$$\mathrm{Crit}_{n,\mathbf{m},k} = \bigcup_{j=1}^{n} \mathrm{Crit}_{n,\mathbf{m},k,j}\,,$$

into subsets $\mathrm{Crit}_{n,\mathbf{m},k,j}$ of generators having x_j as variable of highest index.

Theorem ([JLMO 2026])

The ideal $(M_{n,\mathbf{m},k})$ is minimally generated by the set

$$S_{n,\mathbf{m},k} = \{x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}\} \cup \mathrm{Crit}_{n,\mathbf{m},k}\,.$$

where $x_j^{m_j}$ is removed if $k + \sum_{i=1}^{j-1}(m_i - 1) < m_j$.

Minimal generators and Hilbert series

We partition the set of minimal critical paths,

$$\mathrm{Crit}_{n,\mathbf{m},k} = \bigcup_{j=1}^{n} \mathrm{Crit}_{n,\mathbf{m},k,j} \,,$$

into subsets $\mathrm{Crit}_{n,\mathbf{m},k,j}$ of generators having x_j as variable of highest index.

Theorem ([JLMO 2026])

The ideal $(M_{n,\mathbf{m},k})$ is minimally generated by the set

$$S_{n,\mathbf{m},k} = \{x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}\} \cup \mathrm{Crit}_{n,\mathbf{m},k} \,.$$

where $x_j^{m_j}$ is removed if $k + \sum_{i=1}^{j-1}(m_i - 1) < m_j$.
The Hilbert series of $R/(M_{n,\mathbf{m},k})$ and $R/P_{n,\mathbf{m}}$ satisfy

$$\mathrm{HS}\left(R/(M_{n,\mathbf{m},k}); t\right) = \left[(1 - t^k)\, \mathrm{HS}(R/P_{n,\mathbf{m}}; t)\right] \,,$$

where the brackets denote truncation at the first non-positive coefficient.

The Gröbner basis

Constructing g_s with $\mathrm{in}(g_s) = s$ for every generator s of $(M_{n,\mathbf{m},k})$, obtain:

Theorem ([JLMO 2026])

The reduced Gröbner basis of $I_{n,\mathbf{m},k}$ for $x_1 \succ \cdots \succ x_n$

The Gröbner basis

Constructing g_s with $\mathrm{in}(g_s) = s$ for every generator s of $(M_{n,\mathbf{m},k})$, obtain:

Theorem ([JLMO 2026])

The reduced Gröbner basis of $I_{n,\mathbf{m},k}$ for $x_1 \succ \cdots \succ x_n$ is

$$G_{n,\mathbf{m},k} = \{x_1^{m_1}, \ldots, x_n^{m_n}\} \cup \{g_s \mid s \in \mathrm{Crit}_{n,\mathbf{m},k}\},$$

The Gröbner basis

Constructing g_s with $\mathrm{in}(g_s) = s$ for every generator s of $(M_{n,\mathbf{m},k})$, obtain:

Theorem ([JLMO 2026])

The reduced Gröbner basis of $I_{n,\mathbf{m},k}$ for $x_1 \succ \cdots \succ x_n$ is

$$G_{n,\mathbf{m},k} = \{x_1^{m_1}, \ldots, x_n^{m_n}\} \cup \{g_s \mid s \in \mathrm{Crit}_{n,\mathbf{m},k}\},$$

without $x_j^{m_j}$ if $k + \sum_{i=1}^{j-1}(m_i - 1) < m_j$.

The Gröbner basis

Constructing g_s with $\mathrm{in}(g_s) = s$ for every generator s of $(M_{n,\mathbf{m},k})$, obtain:

Theorem ([JLMO 2026])

The reduced Gröbner basis of $I_{n,\mathbf{m},k}$ for $x_1 \succ \cdots \succ x_n$ is

$$G_{n,\mathbf{m},k} = \{x_1^{m_1}, \ldots, x_n^{m_n}\} \cup \{g_s \mid s \in \mathrm{Crit}_{n,\mathbf{m},k}\},$$

without $x_j^{m_j}$ if $k + \sum_{i=1}^{j-1}(m_i - 1) < m_j$. For $s = s' x_j^{\deg_{x_j}(s)} \in \mathrm{Crit}_{n,\mathbf{m},k,j}$,

$$g_s = \sum_{s'' | s'} \lambda_{s''}\, s''(x_j + \cdots + x_n)^{\deg(s) - \deg(s'')} \mod P_{n,\mathbf{m}},$$

The Gröbner basis

Constructing g_s with $\mathrm{in}(g_s) = s$ for every generator s of $(M_{n,\mathbf{m},k})$, obtain:

Theorem ([JLMO 2026])

The reduced Gröbner basis of $I_{n,\mathbf{m},k}$ for $x_1 \succ \cdots \succ x_n$ is

$$G_{n,\mathbf{m},k} = \{x_1^{m_1}, \ldots, x_n^{m_n}\} \cup \{g_s \mid s \in \mathrm{Crit}_{n,\mathbf{m},k}\},$$

without $x_j^{m_j}$ if $k + \sum_{i=1}^{j-1}(m_i - 1) < m_j$. For $s = s' x_j^{\deg_{x_j}(s)} \in \mathrm{Crit}_{n,\mathbf{m},k,j}$,

$$g_s = \sum_{s'' | s'} \lambda_{s''}\, s''(x_j + \cdots + x_n)^{\deg(s) - \deg(s'')} \mod P_{n,\mathbf{m}},$$

where

$$\lambda_{s''} = \left(\prod_{i=1}^{n-1} \frac{s_i!}{s_i''!}\binom{m_i - s_i'' - 1}{s_i - s_i''}\right) \cdot \frac{s_n!}{(\deg(s) - \deg(s''))!}.$$

Proof idea

The following lemma is shown via an application of inclusion-exclusion.

Lemma ([JLMO 2026])

Let p, q, and r be monomials such that $pq = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $r|pq$. Then

$$\sum_{v \mid \gcd(p,r)} (-1)^{\deg(v)} \left(\prod_{i=1}^{n} \binom{r_i}{v_i} \binom{\alpha_i - v_i}{p_i - v_i} \right) = \prod_{i=1}^{n} \binom{\alpha_i - r_i}{p_i}.$$

Proof idea

The following lemma is shown via an application of inclusion-exclusion.

Lemma ([JLMO 2026])

Let p, q, and r be monomials such that $pq = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $r | pq$. Then

$$\sum_{v |\, \gcd(p,r)} (-1)^{\deg(v)} \left(\prod_{i=1}^{n} \binom{r_i}{v_i} \binom{\alpha_i - v_i}{p_i - v_i} \right) = \prod_{i=1}^{n} \binom{\alpha_i - r_i}{p_i}.$$

Also, we argue as follows.

- The Hilbert series of $R/I_{n,\mathbf{m},k}$ and $R/\operatorname{in}(I_{n,\mathbf{m},k})$ are equal.

Proof idea

The following lemma is shown via an application of inclusion-exclusion.

Lemma ([JLMO 2026])

Let p, q, and r be monomials such that $pq = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $r | pq$. Then

$$\sum_{v | \gcd(p,r)} (-1)^{\deg(v)} \left(\prod_{i=1}^{n} \binom{r_i}{v_i} \binom{\alpha_i - v_i}{p_i - v_i} \right) = \prod_{i=1}^{n} \binom{\alpha_i - r_i}{p_i}.$$

Also, we argue as follows.

- The Hilbert series of $R/I_{n,\mathbf{m},k}$ and $R/\operatorname{in}(I_{n,\mathbf{m},k})$ are equal.
- The Hilbert series of $R/(M_{n,\mathbf{m},k})$ is $\left[(1-t^k)\,\mathrm{HS}(R/P_{n,\mathbf{m}};t)\right]$.

Proof idea

The following lemma is shown via an application of inclusion-exclusion.

Lemma ([JLMO 2026])

Let p, q, and r be monomials such that $pq = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $r | pq$. Then

$$\sum_{v | \gcd(p,r)} (-1)^{\deg(v)} \left(\prod_{i=1}^{n} \binom{r_i}{v_i} \binom{\alpha_i - v_i}{p_i - v_i} \right) = \prod_{i=1}^{n} \binom{\alpha_i - r_i}{p_i}.$$

Also, we argue as follows.

- The Hilbert series of $R/I_{n,\mathbf{m},k}$ and $R/\operatorname{in}(I_{n,\mathbf{m},k})$ are equal.
- The Hilbert series of $R/(M_{n,\mathbf{m},k})$ is $\left[(1-t^k)\operatorname{HS}(R/P_{n,\mathbf{m}};t)\right]$.
- For each minimal generator s of $(M_{n,\mathbf{m},k})$, we have $s \in \operatorname{in}(I_{n,\mathbf{m},k})$.

Proof idea

The following lemma is shown via an application of inclusion-exclusion.

Lemma ([JLMO 2026])

Let p, q, and r be monomials such that $pq = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $r | pq$. Then

$$\sum_{v | \gcd(p,r)} (-1)^{\deg(v)} \left(\prod_{i=1}^{n} \binom{r_i}{v_i} \binom{\alpha_i - v_i}{p_i - v_i} \right) = \prod_{i=1}^{n} \binom{\alpha_i - r_i}{p_i}.$$

Also, we argue as follows.

- The Hilbert series of $R/I_{n,\mathbf{m},k}$ and $R/\operatorname{in}(I_{n,\mathbf{m},k})$ are equal.
- The Hilbert series of $R/(M_{n,\mathbf{m},k})$ is $\left[(1-t^k)\,\mathrm{HS}(R/P_{n,\mathbf{m}};t)\right]$.
- For each minimal generator s of $(M_{n,\mathbf{m},k})$, we have $s \in \operatorname{in}(I_{n,\mathbf{m},k})$.
- Thus $\mathrm{HS}(R/I_{n,\mathbf{m},k};t)$ has $\left[(1-t^k)\,\mathrm{HS}(R/P_{n,\mathbf{m}};t)\right]$ as upper bound.

Proof idea

The following lemma is shown via an application of inclusion-exclusion.

Lemma ([JLMO 2026])

Let p, q, and r be monomials such that $pq = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $r | pq$. Then

$$\sum_{v \mid \gcd(p,r)} (-1)^{\deg(v)} \left(\prod_{i=1}^{n} \binom{r_i}{v_i} \binom{\alpha_i - v_i}{p_i - v_i} \right) = \prod_{i=1}^{n} \binom{\alpha_i - r_i}{p_i}.$$

Also, we argue as follows.

- The Hilbert series of $R/I_{n,\mathbf{m},k}$ and $R/\operatorname{in}(I_{n,\mathbf{m},k})$ are equal.
- The Hilbert series of $R/(M_{n,\mathbf{m},k})$ is $\left[(1-t^k)\operatorname{HS}(R/P_{n,\mathbf{m}};t)\right]$.
- For each minimal generator s of $(M_{n,\mathbf{m},k})$, we have $s \in \operatorname{in}(I_{n,\mathbf{m},k})$.
- Thus $\operatorname{HS}(R/I_{n,\mathbf{m},k};t)$ has $\left[(1-t^k)\operatorname{HS}(R/P_{n,\mathbf{m}};t)\right]$ as upper bound.
- $\left[(1-t^k)\operatorname{HS}(R/P_{n,\mathbf{m}};t)\right]$ is also lower bound for $\operatorname{HS}(R/I_{n,\mathbf{m},k};t)$.

Proof idea

The following lemma is shown via an application of inclusion-exclusion.

Lemma ([JLMO 2026])

Let p, q, and r be monomials such that $pq = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $r | pq$. Then

$$\sum_{v | \gcd(p,r)} (-1)^{\deg(v)} \left(\prod_{i=1}^{n} \binom{r_i}{v_i} \binom{\alpha_i - v_i}{p_i - v_i} \right) = \prod_{i=1}^{n} \binom{\alpha_i - r_i}{p_i}.$$

Also, we argue as follows.

- The Hilbert series of $R/I_{n,\mathbf{m},k}$ and $R/\operatorname{in}(I_{n,\mathbf{m},k})$ are equal.
- The Hilbert series of $R/(M_{n,\mathbf{m},k})$ is $\left[(1-t^k)\operatorname{HS}(R/P_{n,\mathbf{m}};t)\right]$.
- For each minimal generator s of $(M_{n,\mathbf{m},k})$, we have $s \in \operatorname{in}(I_{n,\mathbf{m},k})$.
- Thus $\operatorname{HS}(R/I_{n,\mathbf{m},k};t)$ has $\left[(1-t^k)\operatorname{HS}(R/P_{n,\mathbf{m}};t)\right]$ as upper bound.
- $\left[(1-t^k)\operatorname{HS}(R/P_{n,\mathbf{m}};t)\right]$ is also lower bound for $\operatorname{HS}(R/I_{n,\mathbf{m},k};t)$.
- $\operatorname{HS}(R/I_{n,\mathbf{m},k};t) = \left[(1-t^k)\operatorname{HS}(R/P_{n,\mathbf{m}};t)\right]$ follows.

The special role of the reverse lexicographic order

Remark ([JLMO 2026])

- If $n_1 < n_2$ and $\mathbf{m}_1 \in \mathbb{Z}_{\geq 2}^{n_1}$ is a prefix of $\mathbf{m}_2 \in \mathbb{Z}_{\geq 2}^{n_2}$, then $(M_{n_1,\mathbf{m},k}) \subseteq (M_{n_2,\mathbf{m},k})$.

The special role of the reverse lexicographic order

Remark ([JLMO 2026])

- If $n_1 < n_2$ and $\mathbf{m}_1 \in \mathbb{Z}_{\geq 2}^{n_1}$ is a prefix of $\mathbf{m}_2 \in \mathbb{Z}_{\geq 2}^{n_2}$, then $(M_{n_1,\mathbf{m},k}) \subseteq (M_{n_2,\mathbf{m},k})$.
- Almost reverse lexicographic property: Let $s \in M_{n,\mathbf{m},k}$ be a minimal generator of degree d. Let t be any monomial of degree d such that $t \succ s$ with respect to the reverse lexicographic ordering. Then $t \in (M_{n,\mathbf{m},k})$.

The special role of the reverse lexicographic order

Remark ([JLMO 2026])

- If $n_1 < n_2$ and $\mathbf{m}_1 \in \mathbb{Z}_{\geq 2}^{n_1}$ is a prefix of $\mathbf{m}_2 \in \mathbb{Z}_{\geq 2}^{n_2}$, then $(M_{n_1,\mathbf{m},k}) \subseteq (M_{n_2,\mathbf{m},k})$.
- Almost reverse lexicographic property: Let $s \in M_{n,\mathbf{m},k}$ be a minimal generator of degree d. Let t be any monomial of degree d such that $t \succ s$ with respect to the reverse lexicographic ordering. Then $t \in (M_{n,\mathbf{m},k})$.

Proposition ([JLMO 2026])

(i) *If $x_1 > \cdots > x_n$, then the reduced GB of $I_{n,\mathbf{m},k}$ is $G_{n,\mathbf{m},k}$.*

The special role of the reverse lexicographic order

Remark ([JLMO 2026])

- If $n_1 < n_2$ and $\mathbf{m}_1 \in \mathbb{Z}_{\geq 2}^{n_1}$ is a prefix of $\mathbf{m}_2 \in \mathbb{Z}_{\geq 2}^{n_2}$, then $(M_{n_1,\mathbf{m},k}) \subseteq (M_{n_2,\mathbf{m},k})$.
- Almost reverse lexicographic property: Let $s \in M_{n,\mathbf{m},k}$ be a minimal generator of degree d. Let t be any monomial of degree d such that $t \succ s$ with respect to the reverse lexicographic ordering. Then $t \in (M_{n,\mathbf{m},k})$.

Proposition ([JLMO 2026])

(i) *If $x_1 > \cdots > x_n$, then the reduced GB of $I_{n,\mathbf{m},k}$ is $G_{n,\mathbf{m},k}$.*

(ii) *The ideal $I_{n,\mathbf{m},k}$ has at most $n!$ distinct reduced Gröbner bases, and all of them are degree reverse lexicographic.*

Table of Contents

s-binomial coefficients and s-Catalan triangle

Recall the *s-binomial coefficients* $\binom{n}{d}_s$, which satisfy

$$(1 + x + \cdots + x^s)^n = \sum_{d=0}^{sn} \binom{n}{d}_s x^d \,.$$

s-binomial coefficients and s-Catalan triangle

Recall the *s-binomial coefficients* $\binom{n}{d}_s$, which satisfy

$$(1 + x + \cdots + x^s)^n = \sum_{d=0}^{sn} \binom{n}{d}_s x^d \,.$$

Definition (e.g. [Ghemit and Ahmia 2024, Linz 2022])

The entries of the *s-Catalan triangle* are given by

$$C_{0,0}^{(s)} = 1 \quad \text{and} \quad C_{n,k}^{(s)} = \binom{2n}{sn+k}_s - \binom{2n}{sn+k+1}_s$$

for $n > 0$ and $k \geq 0$.

s-binomial coefficients and s-Catalan triangle

Recall the *s-binomial coefficients* $\binom{n}{d}_s$, which satisfy

$$(1 + x + \cdots + x^s)^n = \sum_{d=0}^{sn} \binom{n}{d}_s x^d \,.$$

Definition (e.g. [Ghemit and Ahmia 2024, Linz 2022])

The entries of the *s-Catalan triangle* are given by

$$C_{0,0}^{(s)} = 1 \quad \text{and} \quad C_{n,k}^{(s)} = \binom{2n}{sn+k}_s - \binom{2n}{sn+k+1}_s$$

for $n > 0$ and $k \geq 0$. The s-Catalan triangle, written with entry $C_{n,k}^{(s)}$ in row n and column k, contains the s-Catalan numbers in its first column.

2-Catalan triangle

We give the first few rows of the 2-Catalan triangle. The 2-Catalan numbers are in its first column.

	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1	1								
2	3	6	6	3	1						
3	15	36	40	29	15	5	1				
4	91	232	280	238	154	76	28	7	1		
5	603	1585	2025	1890	1398	837	405	155	45	9	1

Figure: The 2-Catalan triangle.

2-Catalan triangle

We give the first few rows of the 2-Catalan triangle. The 2-Catalan numbers are in its first column.

	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1	1								
2	3	6	6	3	1						
3	15	36	40	29	15	5	1				
4	91	232	280	238	154	76	28	7	1		
5	603	1585	2025	1890	1398	837	405	155	45	9	1

Figure: The 2-Catalan triangle.

s-Catalan numbers

The *s-Catalan numbers* are given by

$$C_0^{(s)} = 1 \quad \text{and} \quad C_n^{(s)} = \binom{2n}{sn}_s - \binom{2n}{sn+1}_s \text{ for } n > 0.$$

	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8
3	5	15	34	65	111	175	260
4	14	91	364	1085	2666	5719	11096
5	42	603	4269	19845	70146	204687	518498

Figure: The s-Catalan numbers for $n \leq 5$ and $s \leq 7$.

s-Catalan numbers

The *s-Catalan numbers* are given by

$$C_0^{(s)} = 1 \quad \text{and} \quad C_n^{(s)} = \binom{2n}{sn}_s - \binom{2n}{sn+1}_s \text{ for } n > 0.$$

	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8
3	5	15	34	65	111	175	260
4	14	91	364	1085	2666	5719	11096
5	42	603	4269	19845	70146	204687	518498

Figure: The s-Catalan numbers for $n \leq 5$ and $s \leq 7$.

s-Catalan numbers

The *s-Catalan numbers* are given by

$$C_0^{(s)} = 1 \quad \text{and} \quad C_n^{(s)} = \binom{2n}{sn}_s - \binom{2n}{sn+1}_s \text{ for } n > 0.$$

	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8
3	5	15	34	65	111	175	260
4	14	91	364	1085	2666	5719	11096
5	42	603	4269	19845	70146	204687	518498

Figure: The s-Catalan numbers for $n \leq 5$ and $s \leq 7$.

s-Catalan numbers

The *s-Catalan numbers* are given by

$$C_0^{(s)} = 1 \quad \text{and} \quad C_n^{(s)} = \binom{2n}{sn}_s - \binom{2n}{sn+1}_s \text{ for } n > 0.$$

	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8
3	5	15	34	65	111	175	260
4	14	91	364	1085	2666	5719	11096
5	42	603	4269	19845	70146	204687	518498

Figure: The s-Catalan numbers for $n \leq 5$ and $s \leq 7$.

Our approach

We obtain lattice paths for s-Catalan numbers, generalizing Dyck paths for Catalan numbers.

Log-concavity and log-convexity

- A sequence $(a_n)_{n\geq 0}$ of nonnegative real numbers is (e.g. [Zhu 2013])
  - *log-concave* if $a_n^2 \geq a_{n-1}a_{n+1}$ for all $n \geq 1$
  - *log-convex* if $a_n^2 \leq a_{n-1}a_{n+1}$ for all $n \geq 1$
- [Ghemit and Ahmia 2024] $\rightsquigarrow$
  - Rows of the 2-Catalan triangle are log-concave.

Log-concavity and log-convexity

- A sequence $(a_n)_{n\geq 0}$ of nonnegative real numbers is (e.g. [Zhu 2013])
  - *log-concave* if $a_n^2 \geq a_{n-1}a_{n+1}$ for all $n \geq 1$
  - *log-convex* if $a_n^2 \leq a_{n-1}a_{n+1}$ for all $n \geq 1$
- [Ghemit and Ahmia 2024] $\rightsquigarrow$
  - Rows of the 2-Catalan triangle are log-concave.
- Method: certain lattice paths constrained to the first quadrant.

Log-concavity and log-convexity

- A sequence $(a_n)_{n\geq 0}$ of nonnegative real numbers is (e.g. [Zhu 2013])
  - *log-concave* if $a_n^2 \geq a_{n-1}a_{n+1}$ for all $n \geq 1$
  - *log-convex* if $a_n^2 \leq a_{n-1}a_{n+1}$ for all $n \geq 1$
- [Ghemit and Ahmia 2024] $\rightsquigarrow$
  - Rows of the 2-Catalan triangle are log-concave.
- Method: certain lattice paths constrained to the first quadrant.
- Open question in [Ghemit and Ahmia 2024]: Analyze log-convexity/concavity of rows/columns of any s-Catalan triangle.

Log-concavity and log-convexity

- A sequence $(a_n)_{n\geq 0}$ of nonnegative real numbers is (e.g. [Zhu 2013])
  - *log-concave* if $a_n^2 \geq a_{n-1}a_{n+1}$ for all $n \geq 1$
  - *log-convex* if $a_n^2 \leq a_{n-1}a_{n+1}$ for all $n \geq 1$
- [Ghemit and Ahmia 2024] $\rightsquigarrow$
  - Rows of the 2-Catalan triangle are log-concave.
- Method: certain lattice paths constrained to the first quadrant.
- Open question in [Ghemit and Ahmia 2024]: Analyze log-convexity/concavity of rows/columns of any s-Catalan triangle.

Other known results (see [Liu and Wang 2017] for more):

- The sequence of 1-Catalan numbers is log-convex.
- The sequence of Motzkin numbers is log-convex. [Aigner 1998; algebraic proof]; [Callan; combinatorial proof]
- The sequence of 2-Catalan numbers is log-convex [Ghemit and Ahmia 2024].

Log-concavity of the rows

Using our results, we provide the following lattice path interpretation of the numbers in the s-Catalan triangle for any positive integer s.

Proposition ([JLMO 2026])

For $m \geq 2$, $n \geq 1$, and $k \geq 0$, $C^s_{n,k}$ counts the $\mathbf{m}$-admissible lattice paths of degree $ns - k$ whose $\mathbf{m}$-free monomials are not in $\mathrm{in}(I_{2n,\mathbf{m},1})$.

Log-concavity of the rows

Using our results, we provide the following lattice path interpretation of the numbers in the s-Catalan triangle for any positive integer s.

Proposition ([JLMO 2026])

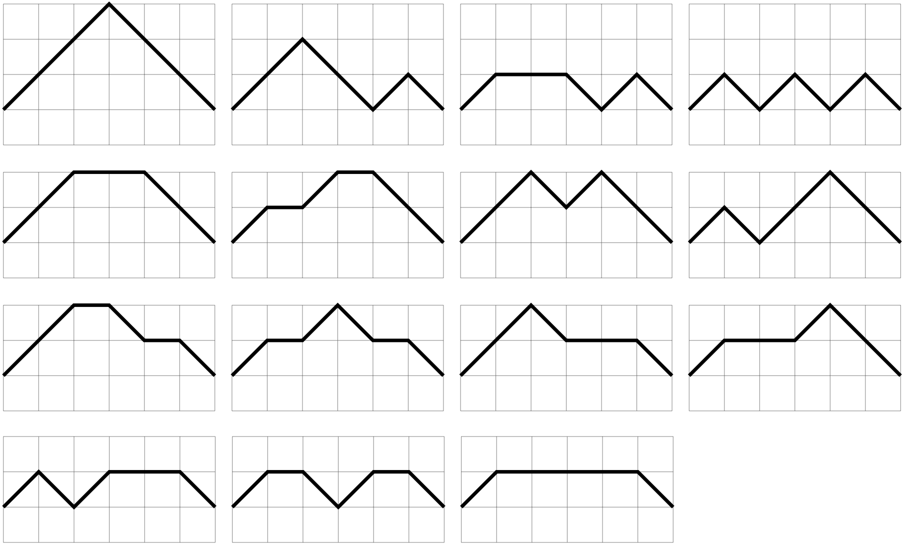
For $m \geq 2$, $n \geq 1$, and $k \geq 0$, $C^s_{n,k}$ counts the $\mathbf{m}$-admissible lattice paths of degree $ns - k$ whose $\mathbf{m}$-free monomials are not in $\mathrm{in}(I_{2n,\mathbf{m},1})$.

Corollary ([JLMO 2026])

The rows of the s-Catalan triangle are log-concave.

Example

We illustrate the lattice paths for $C_3^{(2)} = 15$.



Are s-Catalan numbers log-convex?

We obtain finite sets whose cardinalities are the s-Catalan numbers.

Definition

For $s \geq 1$ and $n \geq 0$, define

$$\mathcal{C}_n^{(s)} = \{\mathbf{m}\text{-admissible non-critical paths ending at } (2n, n\cdot(2-s))\}\,.$$

We have $|\mathcal{C}_n^{(s)}| = C_{n,0}^{(s)}$.

Are s-Catalan numbers log-convex?

We obtain finite sets whose cardinalities are the s-Catalan numbers.

Definition

For $s \geq 1$ and $n \geq 0$, define

$$\mathcal{C}_n^{(s)} = \{\mathbf{m}\text{-admissible non-critical paths ending at } (2n, n \cdot (2-s))\} .$$

We have $|\mathcal{C}_n^{(s)}| = C_{n,0}^{(s)}$.
The problem of deciding the log-convexity of s-Catalan numbers can be formulated as follows.

Open problem

Does there exist an injective map

$$f_n : \mathcal{C}_n^{(s)} \times \mathcal{C}_n^{(s)} \to \mathcal{C}_{n-1}^{(s)} \times \mathcal{C}_{n+1}^{(s)}$$

for every integer $n \geq 1$? If yes, give an explicit construction.

Outlook

- Our critical path ideals can be used to describe
  - Catalan numbers and their convolutions
  - Motzkin and Riordan numbers and their convolutions
  - Entries in s-Catalan triangles
  - ... and many other, also previously undescribed, sequences

Outlook

- Our critical path ideals can be used to describe
  - Catalan numbers and their convolutions
  - Motzkin and Riordan numbers and their convolutions
  - Entries in *s*-Catalan triangles
  - ... and many other, also previously undescribed, sequences
- Relationships to
  - Other generalizations of Catalan numbers
  - Other classes of lattice paths

  remain to be investigated.

Outlook

- Our critical path ideals can be used to describe
  - Catalan numbers and their convolutions
  - Motzkin and Riordan numbers and their convolutions
  - Entries in s-Catalan triangles
  - ...and many other, also previously undescribed, sequences
- Relationships to
  - Other generalizations of Catalan numbers
  - Other classes of lattice paths

  remain to be investigated.
- We are currently exploring what our techniques allow us to say about degree reverse lexicographic Gröbner bases for other types of generic ideals.

Outlook

- Our critical path ideals can be used to describe
  - Catalan numbers and their convolutions
  - Motzkin and Riordan numbers and their convolutions
  - Entries in s-Catalan triangles
  - ...and many other, also previously undescribed, sequences
- Relationships to
  - Other generalizations of Catalan numbers
  - Other classes of lattice paths

  remain to be investigated.
- We are currently exploring what our techniques allow us to say about degree reverse lexicographic Gröbner bases for other types of generic ideals.

Thank you